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On the Gravitational Back Reaction to Hawking Radiation.

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Abstract

We show that a surface term should be added to the Einstein-Hilbert action in order to properly describe quantum transitions occurring around a black hole. The introduction of this boundary term has been advocated by Teitelboim and collaborators and it has been used in the computation of the black hole entropy. Here, we use it to compute the gravitational corrections to the transition amplitudes giving rise to Hawking radiation. This surface term implies that the probability to emit a particle is given by $e^{-\Delta A/4}$ where ΔA is the change in the area of the black hole horizon induced by the emission. Its inclusion at the level of the amplitudes therefore relates quantum black hole radiation to the first law of black hole dynamics. In both cases indeed, the term expressing the change in area directly results from the same boundary term introduced for the same reason: to obtain a well defined action principle.

1 Introduction

There are two possible approaches to the gravitational back reaction to Hawking radiation. The first is to develop a microscopic theory of quantum gravity and to use it to calculate the properties of black holes. This program has been partially realized in the context of super-string theory[1]. The second is to use Hawking's calculation[2] as a starting point to compute gravitational corrections to black hole evaporation. Hopefully these two approaches should meet in some middle ground.

In the more conservative second approach, the back reaction has been addressed along two complementary lines. The first is the “semi-classical” theory wherein the metric remains classical, but one takes as source in Einstein's equations the mean (average) energy-momentum of the second quantized fields propagating on this self consistent background[3][4]. This describes the mean back reaction to Hawking radiation and confirms that black holes evaporate adiabatically in a time $O(M^3)$ as first predicted by Hawking[2].

The second line of attack is to take into account the dynamics of gravity at the level of transition amplitudes before performing any quantum averaging. In the approach initiated by Keski-Vakkuri, Kraus and Wilczek (KKW) [5], one replaces the matter action S_{matter} in a given geometry by the self consistent action of matter plus gravity $S_{matter+gravity}$. One then postulates that the wave functions governing transition amplitudes are WKB, i.e. are of the form $e^{iS_{m+g}}$. This approach closely corresponds to the computation of the transition amplitudes in quantum cosmology performed in [6]. Indeed, the gravitational waves appearing in matrix elements are WKB solutions of the Wheeler-De Witt equation and thus have also their phases determined by the matter energy of the quanta involved in the transition. Another useful analogy to this approach is provided by the quantum description of the trajectory of an accelerated detector experiencing the Unruh effect[7]. In that case, the given trajectory (which plays the role of the classical geometry in black hole physics) has been replaced by WKB waves so as to take into account recoil effects according to Feynman rules[8][9].

The aim of the present work is to clarify the mechanisms at work in the approach of KKW. In particular, we determine what is the correct gravitational action that must be used upon describing black hole radiance. There is indeed some ambiguity in the choice of the boundary terms at the horizon. On physical grounds, we select the appropriate one. This amounts to work with the boost parameter defined at the horizon as the time parameter instead of the time defined at infinity. In this, our treatment of the boundary terms closely follows the work of Bañados, Teitelboim and Zanelli[10].

In a former paper[11], we already used their improved action in a Euclidean context. Our aim was to compute self consistently the action of instantons governing probability transitions in the presence of horizons. The main advantage of the Euclidean approach lies in its simplicity. In that case indeed, the choice of the correct action is particularly transparent. However this approach has an important weakness: one must postulate, as in all works based on the Euclidean path integral see e.g. [12], that no conical singularity be present in the Euclidean geometry. This condition should be proven from first (quantum) principles and not assumed. Remember that in the absence of

gravitational back-reaction it is the fact that the quantum state is vacuum that implies Green functions possessing well defined Euclidean properties.

The present work, that of KKW, and that of [11] all reach the same conclusion. By using the Einstein-Hilbert action rather than only the matter action as the phase of WKB waves, one finds that the rate of emission of particles of energy λ from a black hole of mass M is

$$R_{M \rightarrow M-\lambda} = N(\lambda, M) e^{-\Delta A(\lambda, M)/4} \quad (1)$$

where $\Delta A(\lambda, M) = A(M) - A(M - \lambda)$ is the change of area of the black hole due to the emission of a particle of energy λ , and N is a phase space (also called grey body) factor which cannot be calculated in this approximation scheme. To first order in λ , $\Delta A(\lambda, M)/4 = \lambda/8\pi M$ and one recovers Hawking's canonical result characterized by a temperature.

One of the main interests of eq. (1) is to suggest that the thermal properties of Hawking radiation and the first law of black hole mechanics both stem from the same fundamental principle valid at the quantum level, i.e. before averaging. This would explain why they give a consistent description of the thermal properties of black holes. We recall that the first law relates neighboring black hole classical configurations and as such seems to be completely independent of quantum black hole radiance. A first direct indication of the relationship between the first law and black hole evaporation is provided by the instantonic approach[11]. Indeed in this work, $\Delta A(\lambda, M)/4$ has exactly the same origin as the $A/4$ term in the partition function of a black hole calculated using Euclidean path integral techniques[13][10]. In both these cases, this term arises from a boundary term which is introduced at the horizon in order to select the Hamiltonian variational principle appropriate for the physical problem under investigation. And in the first law as formulated by Wald [14], $dA/4$ is also equal to the variation of the same boundary term at the horizon. At this point it should be noticed that this boundary term does not appear in the calculation of KKW. Our aim is therefore to carefully reformulate the calculation of transition amplitudes to establish that the boundary term must be present right from the beginning.

We finally note that the present procedure applies to all processes occurring in the presence of horizons and leads to transition rates determined by the corresponding changes in area. In particular it also applies to the transitions of accelerated systems. This suggests that there might exist a quantum statistical extension of the thermodynamical approach to general relativity[15] which is valid everywhere and not only applicable to black hole horizons. In this case one would obtain a kind of universal "holographic principle"[16] valid for all horizons and expressing the changes of the effective degrees of freedom to which systems living on one side are coupled.

The plan of the paper is as follows: we begin by reviewing the standard derivations of Hawking radiation without back reaction, following [2] and [7]. We then review the role of boundary terms in the Einstein-Hilbert action. Finally we rederive eq. (1) in two complementary ways. The first emphasizes the role of detectors which measure the presence of particles and which deform differently the geometry according to their quantum state. The second way only makes use of the regular structure of the wave functions near the horizon when expressed in advanced Eddington-Finkelstein coordinates.

2 Hawking Radiation

In this section we fix the notations and review two standard derivations of black hole radiation. In the first, following Unruh's work[7], one introduces a particle detector whose position is fixed and one determines the populations of quanta from its transition rates. In this way, one only uses basic quantum mechanical rules. The second approach is more intrinsic and closer to the original Hawking's derivation[2]. Black hole radiation is established through the Bogoljubov transformation relating in-modes which determine the Heisenberg state and out-modes which define the on-shell particles found at infinity. The specification of the in-modes will be obtained by imposing well defined analytical properties on the horizon[7][17].

The metric of a Schwarzschild black hole is powerful

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \quad (2)$$

We introduce the light like coordinates u and v :

$$v, u = t \pm r^* \quad , \quad r^* = r + 2M|\ln(r - 2M)| \quad (3)$$

and the Kruskal coordinates U and V :

$$U = -\frac{1}{\kappa}e^{-\kappa u} \quad , \quad V = \frac{1}{\kappa}e^{\kappa v} \quad (4)$$

where $\kappa = 1/4M$ is the surface gravity.

For a black hole formed by the collapse of a star, the outgoing modes solution of the Klein Gordon equation have the following form near the horizon:

$$\phi_{\omega,l,m} = \frac{e^{-i\omega U}}{\sqrt{4\pi\omega}} \frac{Y_{lm}(\Omega)}{r} \quad (5)$$

Further from the horizon this is not an exact solution because of the potential barrier which surrounds the black hole. For simplicity in this article we shall neglect the potential barrier. It would only encumber the expressions, whereas all the physics lies near the horizon where it plays no role.

A way to understand the structure of the modes eq. (5) is that they correspond to the absence of particles as seen by an infalling observer. Indeed the proper time lapses of an infalling observer are proportional to ΔU . This is directly apparent by re-expressing the Schwarzschild metric in U, V coordinates. For $r - 2M \ll 2M$, one finds $ds^2 \simeq -dUdV + r^2d^2\Omega$ which explicitizes the initial character of U near $r = 2M$.

The field operator can be decomposed as:

$$\Phi = \sum_{\omega,l,m} a_{\omega,l,m} \phi_{\omega,l,m} + \text{h.c.} + \text{ingoing modes} \quad (6)$$

By definition, the Unruh vacuum state $|0_U\rangle$ is annihilated by the $a_{\omega,l,m}$ operators.

Consider now a particle detector at fixed radius r and angular position Ω which has two levels $|e\rangle$ and $|g\rangle$ of energy E_e and E_g with $\Delta E = E_e - E_g > 0$. Its coupling to the field Φ is given in interaction representation by

$$H_{int} = \gamma \Phi(t, r, \Omega) (e^{+i\Delta Et} |e\rangle\langle g| + \text{h.c.}) \quad (7)$$

where γ is a coupling constant. Initially the detector is in its ground state, hence the combined state of the detector and field is $|0_U\rangle |g\rangle$. In the interacting picture, at late times and to first order in γ , the state of the detector plus field is:

$$\begin{aligned} |\psi_g\rangle &= |0_U\rangle |g\rangle - i\gamma \int dt e^{i\Delta Et} \Phi(r, \Omega, t) |0_U\rangle |e\rangle \\ &= |0_U\rangle |g\rangle - i\gamma \sum_{\omega lm} \int dt e^{i\Delta Et} e^{-i\omega C e^{-\kappa t}} \frac{Y_{lm}(\Omega)}{r\sqrt{4\pi\omega}} a_{\omega lm}^\dagger |0_U\rangle |e\rangle \end{aligned} \quad (8)$$

where $C = (1/\kappa) \exp(\kappa r + \frac{1}{2} \ln(r - 2M))$. Similarly if the detector is initially in its excited state, the amplitude of the detector plus field at late times is

$$|\psi_e\rangle = |0_U\rangle |e\rangle - i\gamma \sum_{\omega lm} \int dt e^{-i\Delta Et} e^{-i\omega C e^{-\kappa t}} \frac{Y_{lm}(\Omega)}{r\sqrt{4\pi\omega}} a_{\omega lm}^\dagger |0_U\rangle |g\rangle \quad (9)$$

The ratio of the amplitude of getting excited to the amplitude of getting deexcited is

$$\frac{\mathcal{A}_{g \rightarrow e, \omega, l, m}}{\mathcal{A}_{e \rightarrow g, \omega, l, m}} = \frac{\int dt e^{+i\Delta Et} e^{-i\omega C e^{-\kappa t}}}{\int dt e^{-i\Delta Et} e^{-i\omega C e^{-\kappa t}}} \quad (10)$$

By replacing t by $t + i\pi/\kappa$ in the upper integral, one obtains that $A_{g \rightarrow e} = A_{e \rightarrow g}^* e^{-\pi \Delta E / \kappa}$, see [18][9]. Thus the ratio of the probabilities of transition is:

$$\frac{|\mathcal{A}_{g \rightarrow e, \omega, l, m}|^2}{|\mathcal{A}_{e \rightarrow g, \omega, l, m}|^2} = e^{-\frac{2\pi}{\kappa} \Delta E} \quad (11)$$

corresponding to the rates in a thermal bath at temperature $\kappa/2\pi = 1/8\pi M$. In equilibrium the ratio of rates is equal to the ratio of probabilities to be in the excited or ground state. This derivation is equivalent to calculating the Bogoljubov transformation between Unruh modes and Schwarzschild modes. Moreover, it provides a physical interpretation to individual Bogoljubov coefficients as transition amplitudes[7][18].

In the second way, the spectrum of emitted particles is determined in terms of modes which are eigenvectors of $i\partial_t = \lambda$. There are two modes for each value of λ which are non vanishing either outside the horizon or inside the horizon:

$$\varphi_{\lambda, l, m, \pm} = \frac{e^{-i\lambda u}}{\sqrt{4\pi\lambda}} \frac{Y_{lm}(\Omega)}{r} \theta(\pm(r - 2M)) \quad (12)$$

Where we have once more neglected the potential barrier outside the black hole. The Schwarzschild modes φ_+ are those which would naturally be used by a static observer around the black hole to describe the presence or absence of particles. On the other

hand the modes $\phi_{\lambda,l,m}$ associated to the Unruh vacuum state are linear combinations of φ_+ and φ_- . To determine this linear combination one must impose that the $\phi_{\lambda,l,m}$ have only positive frequency for an infalling observer. One way to construct the ϕ_λ modes is to express the Schwarzschild modes in Kruskal coordinates: $\varphi_{\lambda\pm} \simeq (\mp U)^{i\lambda/\kappa}$. Since ΔU is proportional to the proper time of an infalling observer, ϕ_λ must be the linear combination of $\varphi_{\lambda\pm}$ which is analytic and bounded in the lower half of the complex U plane:

$$\phi_{\lambda,l,m} = \frac{1}{\sqrt{1 - e^{-2\pi\lambda/\kappa}}} \left(\varphi_{\lambda,l,m,+} + e^{-\pi\lambda/\kappa} \varphi_{\lambda,l,m,-} \right) \quad (13)$$

where the normalization factor ensures that the ϕ modes have unit norm. The weights in eq. (13) define the Bogoljubov coefficients α_λ and β_λ . Their ratio satisfies

$$\frac{|\beta_\lambda|^2}{|\alpha_\lambda|^2} = e^{-2\pi\lambda/\kappa} \quad (14)$$

This implies that the Unruh vacuum is a thermal distribution of Schwarzschild particles at temperature $\kappa/2\pi M$ (compare with eq. (11)).

We will find it convenient below to use the same argument, but rephrased in Eddington-Finkelstein coordinates v, r, Ω in which the metric has the form:

$$ds^2 = -(1 - 2M/r)dv^2 + 2dvdr + r^2 d\Omega^2 \quad (15)$$

Near the horizon the metric takes the simple form $ds^2 \simeq 2dvdr + r^2 d\Omega^2$ which shows that v, r are inertial coordinates near the horizon. An infalling observer follows, near the horizon, approximately the trajectory $v = \text{const}$, so that $-r$ is approximately a proper time parameter along his world line. $\phi_{\lambda,l,m}$ must have only positive frequency in $-r$ at fixed v . Near the horizon, in Eddington-Finkelstein coordinates, the φ modes have the form $\varphi_{\lambda,l,m,\pm} \simeq e^{-i\frac{\Delta}{\kappa}(\kappa v - \ln(r-2M))} \theta(\pm(r-2M))$. The regularity condition imposes that to obtain a ϕ mode, one analytically continues φ_+ in the upper half complex r plane to obtain the relative weight of the φ_+ and φ_- modes, as shown in [17]. In this way one recovers eq. (13).

3 Boundary terms in the Einstein-Hilbert Action

In this section we examine the boundary terms in the Einstein-Hilbert action. We start our analysis with the action expressed in Hamiltonian form:

$$S = \int dt \left\{ \pi^{ab} \dot{g}_{ab} + p\dot{q} - NH - N^i H_i \right\} \quad (16)$$

where g_{ab} is the spatial metric, π^{ab} its conjugate momentum, q and p the coordinates and momentum of matter, N and N^i the lapse and shift, and H and H_i the energy and momentum constraints.

We shall consider this action defined in the right quadrant of the full Kruskal manifold, and take the time foliation to resemble Schwarzschild time. The equal time slices

then have two boundaries: one at spatial infinity and one at the horizon (more precisely at the intersection of the past and future horizon). Variations of the action eq. (16) gives rise to boundary terms because of the presence of spatial derivatives of the metric in H and H_i . In order that the action be extremal on the solutions of the equations of motion, one must add surface terms to S that cancel the boundary terms. We now review these boundary terms, starting with the boundary at infinity. Details of the calculations will not be presented. They can be found in many papers, see for instance [19][20][10][21].

Varying eq. (16) while imposing that the space time is asymptotically flat yields

$$\delta S = \int dt \text{ terms giving eqt. of motion} + t_\infty \delta M_{ADM} \quad (17)$$

where $t_\infty = \int dt N$ is the proper time at infinity and M_{ADM} is the mass at infinity (defined in terms of the behavior of g_{rr} for large r). In this form the action is extremal on the equations of motion if one varies among the class of metrics for which M_{ADM} is kept fixed.

On the contrary, if one varies among the class of metrics for which t_∞ is kept fixed but M_{ADM} is arbitrary one must subtract the surface term $M_{ADM} t_\infty$ to the action S so that the variation of the new action S' yields

$$\delta S' = \int dt \text{ terms giving eqt. of motion} - M_{ADM} \delta t_\infty \quad (18)$$

This second form of the action is generally more convenient because one can calculate S' as a function of t_∞ , and then use this expression to calculate the time evolution of the matter and metric. Indeed since M_{ADM} is a conserved quantity, $\partial S' / \partial M_{ADM} = \text{const}$ gives the time evolution. A second more physical reason is that one may want to compare the evolution of systems with different ADM masses, and hence one does not want to fix it to start with. However we must emphasize that there is nothing fundamentally wrong with the variational principle based on S . This will be important below.

We now turn to the boundary term at the horizon. The analysis proceeds exactly in parallel with the proceeding one. The form of the boundary term is dictated by the fact one requires that near the horizon the lapse and shift vanish (or equivalently that at $\rho = 0$, the momentum π_ρ^ρ and the derivative of the area of the surfaces of constant ρ , $\partial_\rho A$, vanish, see [21]) and hence the metric can be put in the form

$$ds^2 = -\rho^2 \kappa^2 dt^2 + d\rho^2 + r^2(\rho) d\Omega^2 \quad (19)$$

For a Schwarzschild black hole $\rho = \sqrt{8M(r - 2M)}$, $\kappa = 1/4M$ and $t = t_\infty$.

Upon varying eq. (16) one then finds a boundary term at the horizon

$$\delta S = \int dt \text{ terms giving eqt. of motion} - \Theta \delta A / 8\pi \quad (20)$$

where $\Theta = \int \kappa dt$ is the hyperbolic angle and A is the area of the horizon. The action in this form is therefore extremal on the equations of motion for the class of metrics which have fixed horizon area.

One can add to this action a surface term at the horizon and define $S'' = S + \Theta A/8\pi$ so that its variation takes the form:

$$\delta S'' = \int dt \text{ terms giving eqt. of motion} + A\delta\Theta/8\pi \quad (21)$$

This action is extremal on the equations of motion provided one makes variations in the class of metrics which have fixed Θ at the horizon.

This is the action that must be used when calculating the action of the Euclidean continuation of the black hole[10]. Indeed regularity of the Euclidean manifold at the horizon imposes that the Euclidean angle $\Theta_E = 2\pi$. The surface term in the action then contributes a term $A/4$ to the partition function which is interpreted as the entropy of the black hole. This is also the action that was used in [11] to calculate the self-consistent actions of instantons. In the present paper, we shall use it in the Lorentzian sector. The new aspect brought in by its use is that the time evolution will be given in terms of the boost parameter Θ rather than the time at infinity.

Thus upon considering processes occurring around a black hole, there are a priori 4 actions that can be considered according to choice of the surface terms. Two however are unphysical. Indeed fixing both the ADM mass and the horizon area is inconsistent. For instance for the vacuum spherically symmetric solutions, fixing the ADM mass determines the horizon area. Similarly fixing both the time at infinity and the hyperbolic angle Θ is inconsistent. Thus one is left with two possibilities: fixing the ADM mass and Θ or fixing t_∞ and A . A more mathematical reason why these are the only two possibilities is that the constraints $H = 0$ and $H_i = 0$ viewed as differential equations need boundary conditions in order to yield a unique solution and fixing the ADM mass or the horizon area but not both provides the required boundary data[21].

The choice among these two possibilities is dictated by physical considerations. If the ADM mass is not fixed but A is, this means that one is in effect considering situations in which there are exchanges of energy between the matter surrounding the black hole and infinity while leaving the black hole itself unchanged. On the other hand if one fixes M_{ADM} while letting A vary, one is comparing situations in which the black hole and the surrounding matter exchange energy, but no energy is exchanged with infinity.

Clearly the process of black hole evaporation belongs to the second situation. Therefore in the next section we shall consider the action

$$S = \int dt \left\{ \pi^{ab} \dot{g}_{ab} + p\dot{q} - NH - N^i H_i \right\} + \Theta A/8\pi \quad (22)$$

This action gives the time evolution of matter as a function of Θ for different values of A with M_{ADM} fixed.

4 Gravitational Back Reaction to Hawking Radiation

In this section we calculate the new expressions of the different wave functions introduced in section 2 by replacing the matter action in the given Schwarzschild geometry by the

gravitational action eq. (22). We then use these wave functions to compute the gravitational corrections to black hole radiation.

We start with the description of black hole radiation based on the readings of a static detector. It is through the change of the deformation of the geometry induced by the change of the quantum state of the detector that the change in area will appear in the transition rates.

We first compute the new time dependence of the wave functions associated with the two states of the detector. Since the detector is at $r = \text{const}$, the geometry is static and the $p\dot{q}$ and $\pi^{ab}\dot{g}_{ab}$ terms in the action vanish. Moreover, on-shell, the constraints also vanish. Hence the only term contributing to the total action (matter + gravity) is the surface term at the horizon. This term is equal to $\Theta A(g)/8\pi$ or $\Theta A(e)/8\pi$ where $A(g)$ ($A(e)$) is the horizon area when the detector is in the ground (excited) state. Thus the time dependence of the wave functions are

$$\begin{aligned}\psi_g(\Theta) &= e^{i\Theta A(g)/8\pi} \\ \psi_e(\Theta) &= e^{i\Theta A(e)/8\pi}\end{aligned}\tag{23}$$

It remains to determine the new expression for the outgoing modes which replaces eq. (5). As in the absence of back-reaction, the structure of the modes must be such that they determine the vacuum near the horizon. Since Θ has been defined at the horizon, its relationship to the inertial light like coordinates U_{hor}, V_{hor} also defined at the horizon is of the form $U_{hor} = -e^{-\Theta}$. This encodes the exponential Doppler shift which is the hall mark of horizons. Therefore, the new expression is

$$\phi_\omega(\Theta, r) = D e^{iC\omega e^{-\Theta}}\tag{24}$$

in the place of $e^{iC\omega e^{-\kappa t}}$ see eqs. (8, 9).

As in Section 2, the transition amplitudes are given by the “time” integral of the product of the three waves. Up to the *same* overall constant, they are given by

$$\begin{aligned}\mathcal{A}_{g \rightarrow e + \omega} &= \int d\Theta \psi_g(\Theta) \psi_e(\Theta)^* \phi_\omega(\Theta)^* \\ \mathcal{A}_{e \rightarrow g + \omega} &= \int d\Theta \psi_g(\Theta)^* \psi_e(\Theta) \phi_\omega(\Theta)^*\end{aligned}\tag{25}$$

Note that the area of the horizon decreases if the detector is in its excited state. Therefore, in the limit of small ΔE , i.e. in the test particle limit, the factor $\psi_g(\Theta)^* \psi_e(\Theta) = e^{i\Theta(A(e) - A(g))/8\pi}$ tends to the background field expression $e^{-it\Delta E}$ with the correct sign of the phase.

Using the new expressions eq. (25), by replacing Θ by $\Theta + i\pi$ in the upper amplitude, one obtains

$$\frac{|\mathcal{A}_{g \rightarrow e + \omega}|^2}{|\mathcal{A}_{e \rightarrow g + \omega}|^2} = e^{-(A(e) - A(g))/4}\tag{26}$$

It is thus $(A(e) - A(g))/4$, the difference of the horizon areas if the detector is in its excited or ground state which governs the equilibrium distribution of the detector’s states. This describes a micro-canonical distribution since we are considering exchanges

of energy between the black hole and the detector with no mass exchange at infinity. This distribution replaces the canonical expression of eq. (11), governed by the energy change $E_e - E_g$ and Hawking's temperature $\kappa/2\pi$.

The new ratio of the transition rates is a function of the change in area because the integrand of the transition amplitude contained the product $\psi_g^* \psi_e$ which describes the replacement of one classical solution by the other one. It is indeed through this $\psi^* \psi$ product that the change in area entered into the expression since its phase is given by the *difference* of the total matter+gravity actions. As noted above this difference tends to $-\Delta E t$ when ΔE is small. This recovery of the background field wave functions in the limit of small differences of matter energy is a generic feature[22] of taking into account a neglected degree of freedom (here gravity). The same mechanism arises upon taking into account recoil effects in scattering amplitudes, see [8]. It also explains why one recovers the conventional expressions of transition amplitudes starting from the solutions of the Wheeler-DeWitt equation in quantum cosmology[6].

In this calculation the change in area is due to the change of the quantum state of the detector. However the existence of a detector is not intrinsic to black hole radiation: the detector was only used to reveal the existence of the quanta. Therefore we seek for an intrinsic derivation of black hole radiance in which the change in area is due to the emission process itself. To this end, we must introduce a model for the matter waves which takes into account the deformation of the gravitational background. The simplest model is that of KKW in which the matter is described as a spherically symmetric (s-wave) light-like shell. The main result of [5] is that outside and inside the shell the metric is Schwarzschild, but with the mass parameter M_{ADM} and $M_{ADM} - \lambda$ respectively where λ is the energy of the shell. Then the time parameters inside and outside the shell are different and related by junction conditions on the shell. In what follows this relationship will play no role. The shell follows a light like geodesic in both the inside and outside metric. This property allows us to calculate the action of the shell in a straightforward manner.

Inside the shell the metric is

$$ds^2 = -(1 - 2M'/r)dt^2 + (1 - 2M'/r)^{-1}dr^2 + d\Omega^2 \quad (27)$$

where $M' = M_{ADM} - \lambda$ is the final mass of the black hole. The shell follows a geodesic in this metric.

$$\begin{aligned} \frac{dr_{sh}}{dt} &= (1 - 2M'/r_{sh}) \\ t_{sh}(r, M') &= r + 2M' \ln(r - 2M') = r + \frac{1}{2\kappa'} \ln(r - 2M') \end{aligned} \quad (28)$$

where $\kappa' = 1/4M'$.

As in the previous treatment, it is particularly convenient to reexpress the evolution in terms of the hyperbolic angle Θ using the Jacobian $d\Theta/dt = \kappa'$:

$$\Theta_{sh}(r, M') = \kappa' r + \frac{1}{2} \ln(r - 2M') \quad (29)$$

Notice that close to the horizon, the description of the shell's trajectory is now expressed only in terms of quantities locally defined, i.e. insensitive to the matter distribution at larger radii.

The action of the shell can in principle be calculated by solving the Hamilton-Jacobi equations. Since A' is a constant of motion, these can be solved by separation of variables to yield the form $S(A', \Theta) = A'\Theta/8\pi + f(r, A')$. Furthermore $\partial S/\partial A' = 0$ must give the equations of motion which implies that

$$\frac{\partial f}{\partial A'} = -\frac{\Theta_{sh}(r, M')}{8\pi} \quad (30)$$

Inserting eq. (29), this can be integrated to yield

$$f(A') = \int_{A'}^A \frac{d\tilde{A}}{8\pi} \left(\frac{1}{2} \ln(r - r_{hor}(\tilde{A})) + \kappa(\tilde{A})r \right) \quad (31)$$

where we have introduced $r_{hor}(A)$, the radius of the horizon parameterized by its area. Similarly we have introduced $\kappa(A)$. Note however that κ is not strictly speaking the surface gravity of the black hole, rather it is determined by the form of the metric near the horizon eq. (19).

In order to obtain the action which governs the *transition* from A to A' , we must subtract from $S(A')$ the action $A\Theta/8\pi$ which describes the geometry in the absence of the shell. This yields

$$S_{shell}(A', A) = -\Theta(A - A')/8\pi + \int_{A'}^A \frac{d\tilde{A}}{16\pi} \ln(r - r_{hor}(\tilde{A})) + \dots \quad (32)$$

where we have dropped the non logarithmic term which is unimportant near the horizon. (Note that $\Delta_\lambda A = A - A'$ is always positive). Upon postulating that the wave functions are WKB, the Schwarzschild mode describing the shell is $\varphi_{\lambda,+} \simeq e^{iS(A', A)}$ when the gravitational interaction is taken into account

Our task is to now calculate the Bogoljubov transformation between Unruh modes and Schwarzschild modes. To impose the regularity of the Unruh modes at the horizon we use the Eddington-Finkelstein coordinates inside the shell $v = t + (\frac{1}{2\kappa'} \ln(r - r_{hor}) + r)$ and r , and then impose analyticity in the upper complex r plane. In these coordinates eq. (32) becomes

$$S_{shell} = -\Delta_\lambda A \kappa' v / 8\pi + \frac{\Delta_\lambda A}{16\pi} \ln(r - 2M') + \int_{A'}^A \frac{d\tilde{A}}{16\pi} \ln(r - r_{hor}(\tilde{A})) \quad (33)$$

up to non logarithmic terms. Analyticity in the lower half plane imposes that the action on the right differs from the action on the left by taking $r - 2M' \rightarrow (r - 2M')e^{i\pi}$, hence $\ln(r - 2M') \rightarrow \ln(r - 2M') + i\pi$. Since both the second and the third term equally contribute, we obtain

$$S_{left} = S_{right} + i(\Delta_\lambda A/8) \quad (34)$$

The Bogoljubov transformation between Unruh and Schwarzschild modes is therefore

$$\phi_\lambda = N(\lambda)(\varphi_{\lambda,+} + e^{-\Delta_\lambda A/8} \varphi_{\lambda,-}) \quad (35)$$

corresponding to the Bogoljubov coefficients

$$\frac{|\beta_\lambda|^2}{|\alpha_\lambda|^2} = e^{-\Delta_\lambda A/4} \quad (36)$$

and one recovers eq. (1).

Note that the two derivations lead to the same characterization of the vacuum at the horizon. There is indeed a linear combination of *positive* frequency Unruh modes ϕ_λ eq. (35) which at fixed r has the Θ dependence of the form $De^{iC\omega e^{-\Theta}}$.

5 Discussion

We have shown that upon taking into account the gravitational back reaction, the probability for a black hole to emit a quantum is given by the exponential of the change in area of the black hole. The appearance of this factor ΔA has the same origin as the appearance of the term dA in the first law of black hole mechanics, namely the surface term $\Theta A/8\pi$ at the horizon. This derivation is not only valid for Schwarzschild black holes. Indeed both derivations are also valid for charged black holes and should be easily generalizable to rotating black holes. Furthermore the first derivation which does not make appeal to any symmetry also applies to the transitions of uniformly accelerated detectors in Minkowski space (the Unruh effect).

In obtaining this result we have made an assumption about the physics at Planckian scales: we have postulated that the specification of the vacuum state can be implemented by the usual condition of regularity at the horizon when expressed in terms of the *local* coordinates Θ , U_{hor} and r . In other words, we have assumed that the back-reaction does not destroy the usual analytical characterization of the vacuum. One would hope to have a justification for this assumption at a more fundamental level. This probably requires a microscopic description of the physics at the horizon at the Planckian scale, either string theory or some other quantum theory of gravity.

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